Facoltà di Ingegneria Civile e Industriale
Ambiente e Territorio, Sicurezza

Scienza delle Costruzioni

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SAPIENZA
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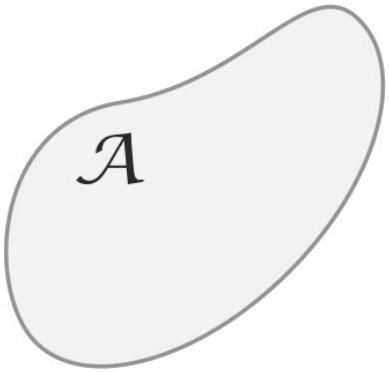
Lezione

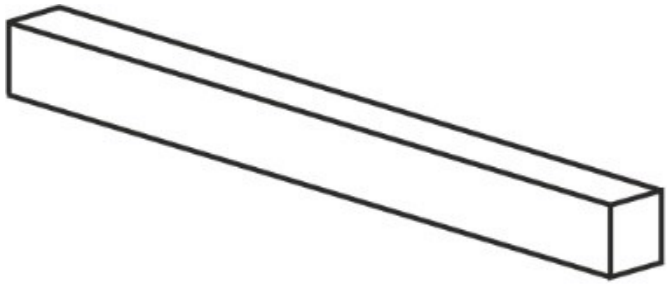
1. Geometria delle Aree

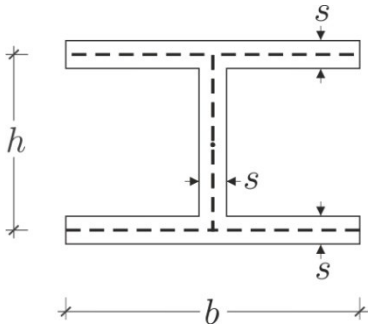


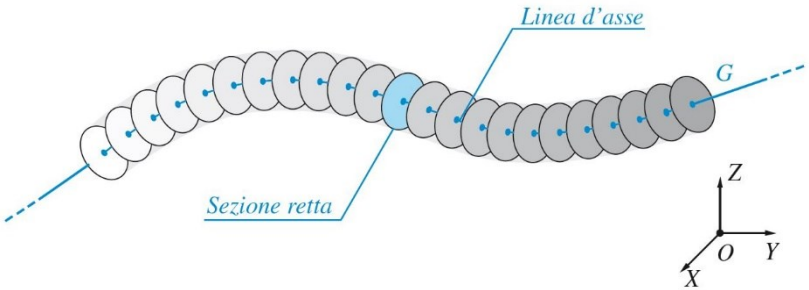
- **Obiettivi**
- **Area e Momenti statici**
- **Centro di figura (baricentro)**
- **Momenti d'inerzia, tensore d'inerzia, momento polare**
- **Assi e Momenti principali d'inerzia, raggi d'inerzia**
- **Formule di trasporto e rotazione**
- **Esercizi** (Esercitazioni E15a-E15b; testo: §A.7-A.9)

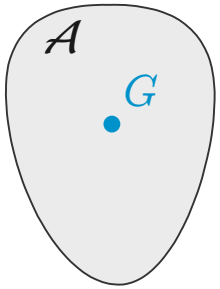
1. Geometria delle aree

Obiettivi: definire le caratteristiche geometriche delle sezioni di trave



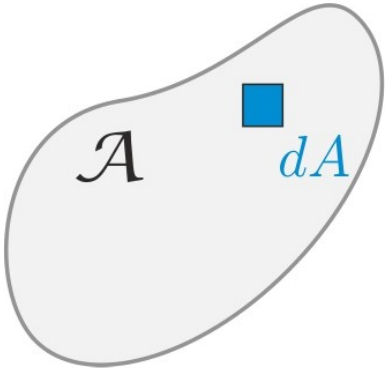






1. Geometria delle aree

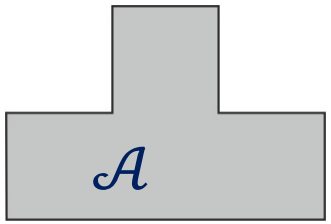
Area

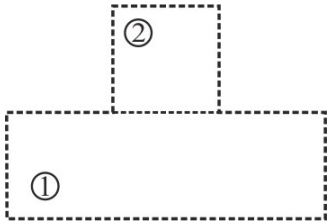


$$A = \int_{\mathcal{A}} dA \qquad [A] = [L^2]$$

$$A > 0$$

Additività



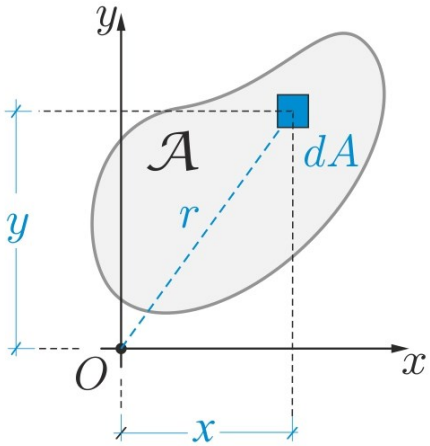


$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \quad \Rightarrow A = A_1 + A_2$$

$$\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i \Rightarrow \quad A = \sum_{i=1}^{n} A_i$$

1. Geometria delle aree

Momento statico rispetto ad un asse (momento del I ordine)



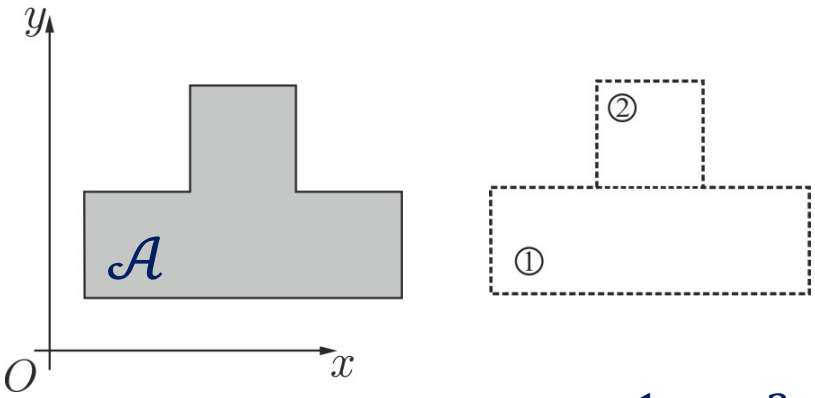
$$dS_x = y\,dA \qquad S_x = \int_{\mathcal{A}} y dA$$

$$dS_y = x\,dA \qquad S_y = \int_{\mathcal{A}} x dA$$

$$[S] = [L^3]$$

$$S \lesseqgtr 0$$

Additività

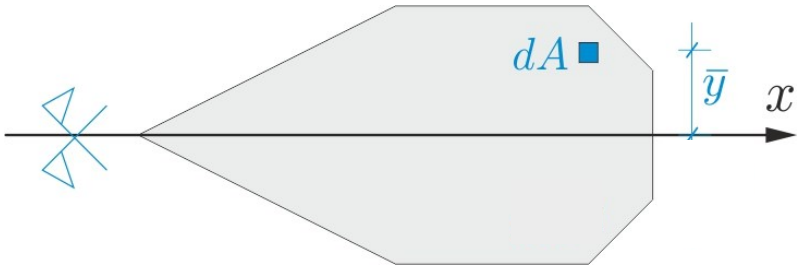


$$\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i \Rightarrow \quad S_x = \sum_{i=1}^{n} S_x^i$$

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \;\Rightarrow S_x = S_x^1 + S_x^2$$

1. Geometria delle aree

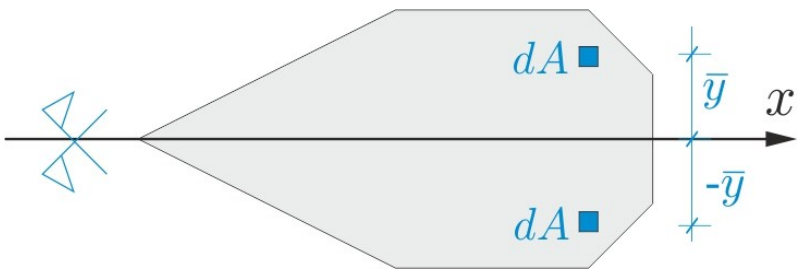
Momento statico rispetto ad un asse di simmetria, ad es. x



$$dS_x = \bar{y} dA$$

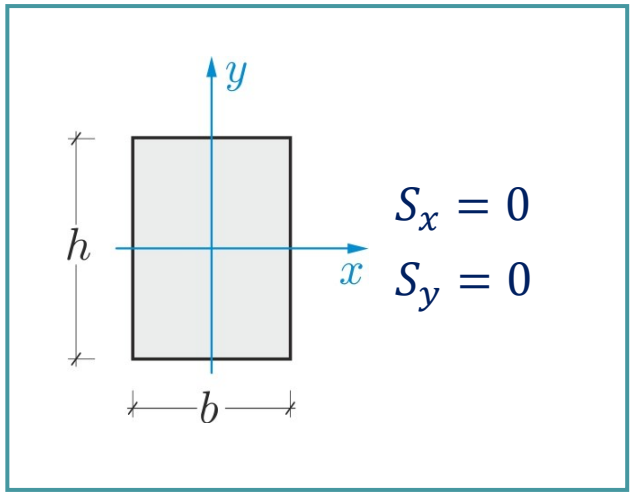
1. Geometria delle aree

Momento statico rispetto ad un asse di simmetria, ad es. x



$$dS_x = \bar{y}dA \qquad S_x = \int_{\mathcal{A}} ydA = 0$$

$$dS_x = -\bar{y}dA \qquad \boxed{S_x = 0}$$



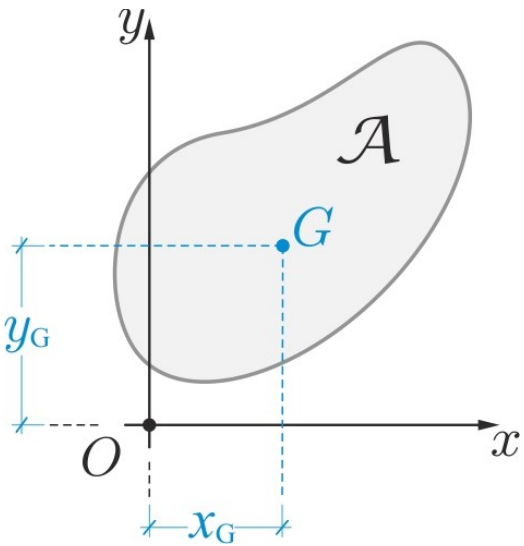
$$S_x = 0$$
$$S_y = 0$$

Osservazione 1

Se la figura piana $\mathcal{A}$ è simmetrica, Il suo momento statico rispetto all'asse di simmetria è nullo

1. Geometria delle aree

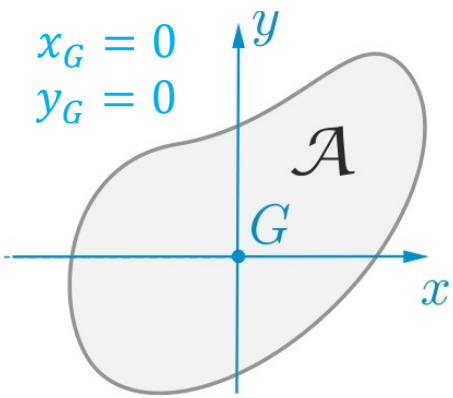
Centro di figura



$$x_G = \frac{\int_{\mathcal{A}} x dA}{\int_{\mathcal{A}} dA} = \frac{S_y}{A} \qquad \Rightarrow S_y = A x_G$$

$$y_G = \frac{\int_{\mathcal{A}} y dA}{\int_{\mathcal{A}} dA} = \frac{S_x}{A} \qquad \Rightarrow S_x = A y_G$$

Assi centrali o baricentrici



$$S_x = A y_G = 0$$

$$S_y = A x_G = 0$$

Osservazione 2

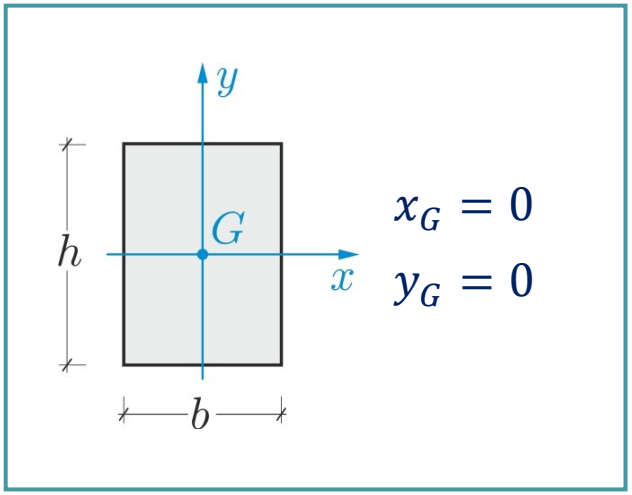
Se un asse è centrale, il momento statico di $\mathcal{A}$ rispetto a tale asse è nullo

Osservazione 3

Se un asse è di simmetria per $\mathcal{A}$, tale asse passa per il baricentro: gli assi di simmetria sono anche centrali

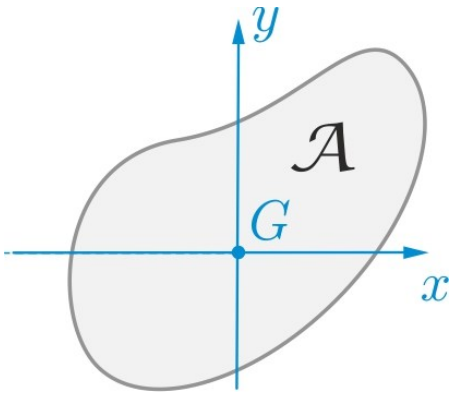
1. Geometria delle aree

Esempio



$$x_G = 0$$
$$y_G = 0$$

Assi centrali o baricentrici



$$S_x = Ay_G = 0$$
$$S_y = Ax_G = 0$$

Osservazione 2

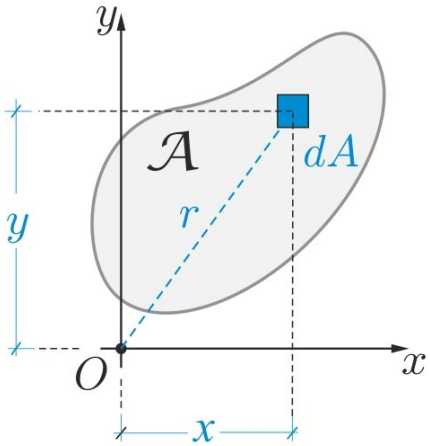
Se un asse è centrale, il momento statico di $\mathcal{A}$ rispetto a tale asse è nullo

Osservazione 3

Se un asse è di simmetria per $\mathcal{A}$, tale asse passa per il baricentro: gli assi di simmetria sono anche centrali

1. Geometria delle aree

Momenti d'inerzia (momenti del II ordine)



$$dI_x = y^2 dA \qquad I_x = \int_{\mathcal{A}} y^2 dA > 0$$

$$dI_y = x^2\, dA$$

$$dI_{xy} = xy dA \qquad I_y = \int_{\mathcal{A}} x^2 dA > 0$$

$$[I] = [L^4]$$

$$I_{xy} = \int_{\mathcal{A}} xy dA \lesseqgtr 0 \qquad \boxed{I_{xy} = I_{yx}}$$

Momento d'inerzia polare

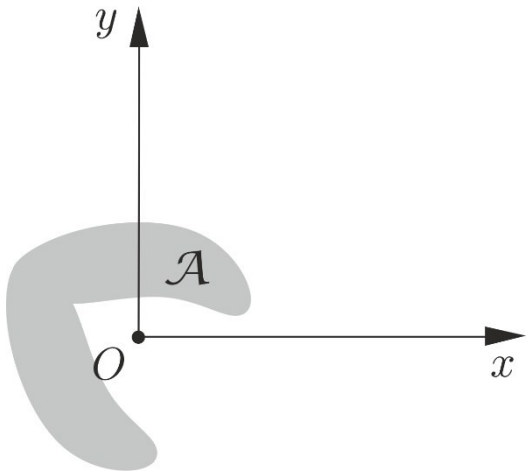
$$dI_O = r^2\, dA = (x^2 + y^2) dA \qquad I_O = I_r = \int_{\mathcal{A}} r^2 dA = I_x + I_y$$

Tensore d'inerzia

$$\mathcal{I} = \begin{bmatrix} \int_{\mathcal{A}} y^2 dA & \int_{\mathcal{A}} xy\, dA \\ \int_{\mathcal{A}} yx\, dA & \int_{\mathcal{A}} x^2 dA \end{bmatrix} = \begin{bmatrix} I_x & I_{xy} \\ I_{xy} & I_y \end{bmatrix}$$

1. Geometria delle aree

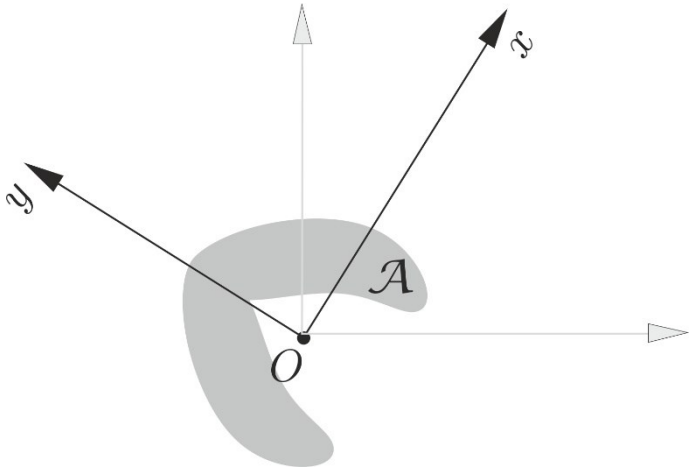
Tensore d'inerzia



$$I_{xy} \neq 0$$

$$\mathcal{I} = \begin{bmatrix} I_x & I_{xy} \\ I_{xy} & I_y \end{bmatrix}$$

Tensore principale d'inerzia



$$I_{xy} = 0$$

$$\mathcal{I} = \begin{bmatrix} I_x & 0 \\ 0 & I_y \end{bmatrix}$$

Assi principali d'inerzia:

x, y

Momenti principali d'inerzia:

I_x, I_y

Se $O \equiv G$:

x, y

Assi centrali principali d'inerzia

1. Geometria delle aree

Momenti e assi principali d'inerzia

$$\boxed{I_{xy} = \int_{\mathcal{A}} xy dA = 0}$$

$$I_x = \int_{\mathcal{A}} y^2 dA > 0$$

$$I_y = \int_{\mathcal{A}} x^2 dA > 0$$

$$[I] = [L^4]$$

Raggi principali d'inerzia

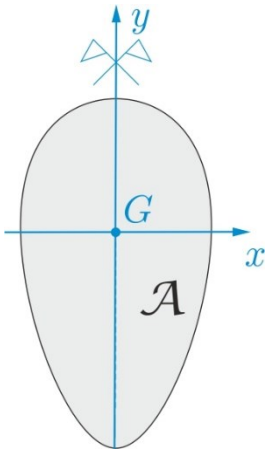
$$\rho_x = \sqrt{\frac{I_x}{A}} \qquad \rho_y = \sqrt{\frac{I_y}{A}} \qquad [\rho] = [L]$$

Tensore principale d'inerzia

$$\mathcal{J} = \begin{bmatrix} I_x & 0 \\ 0 & I_y \end{bmatrix}$$

1. Geometria delle aree

Momenti e assi principali d'inerzia in figure con assi di simmetria



$$\boxed{I_{xy} = \int_{\mathcal{A}} xy\,dA = 0}$$

$$I_x = \int_{\mathcal{A}} y^2 dA > 0$$

$$I_y = \int_{\mathcal{A}} x^2 dA > 0$$

$$[I] = [L^4]$$

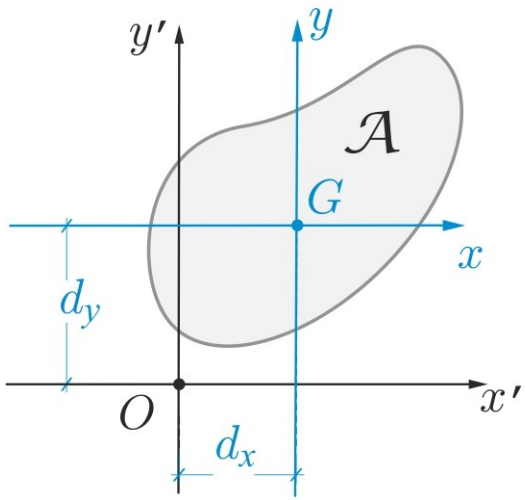
Osservazione 4

Se uno o entrambi gli assi x, y sono di simmetria per $\mathcal{A}$, allora x e y sono principali d'inerzia.

1. Geometria delle aree

Formule di trasporto (Huyghens)

Determinare $I_{x'}$, $I_{y'}$, $I_{x'y'}$
in funzione di I_x, I_y, I_{xy}



$$x' = x + d_x$$
$$y' = y + d_y$$

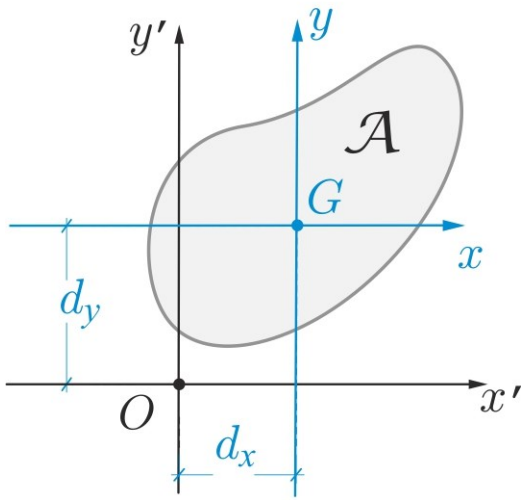
$$I_{x'} = \int_{\mathcal{A}} y'^2 dA = \int_{\mathcal{A}} (y + d_y)^2 dA = \int_{\mathcal{A}} (y^2 + d_y^2 + 2y d_y) dA$$

$$I_{x'} = \int_{\mathcal{A}} y^2 dA + d_y^2 \int_{\mathcal{A}} dA + 2d_y \int_{\mathcal{A}} y\, dA = I_x + d_y^2 A + 2d_y \cancel{S_x} = 0$$

$$\boxed{I_{x'} = I_x + d_y^2 A}$$

1. Geometria delle aree

Formule di trasporto (Huyghens)



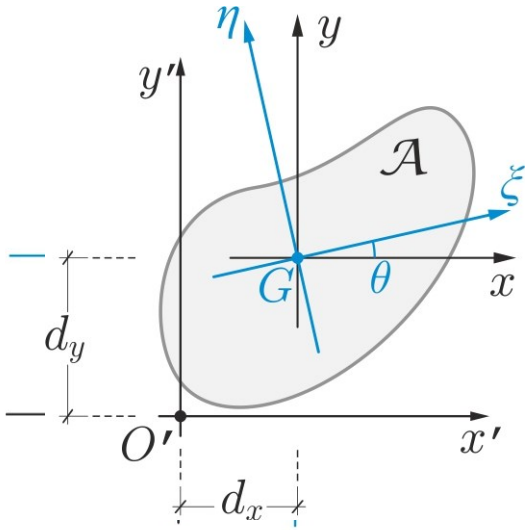
$$I_{x'} = I_x + d_y^2 A$$

$$I_{y'} = I_y + d_x^2 A$$

$$I_{x'y'} = I_{xy} + d_x\, d_y\, A$$

1. Geometria delle aree

Formule di Rotazione



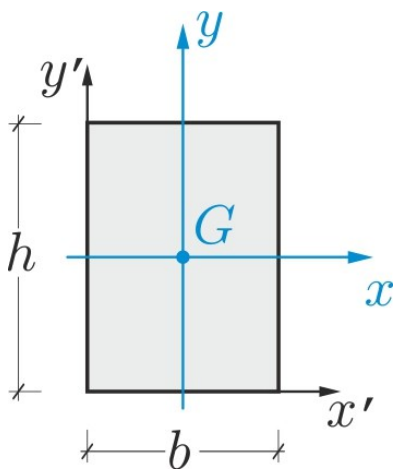
$$I_\xi = I_x \cos^2\theta + I_y \sin^2\theta - 2I_{xy}\cos\theta\sin\theta$$

$$I_\eta = I_x \sin^2\theta + I_y \cos^2\theta + 2I_{xy}\cos\theta\sin\theta$$

$$I_{\xi\eta} = I_{xy}(\cos^2\theta - \sin^2\theta) + (I_x - I_y)\cos\theta\sin\theta$$

1. Geometria delle aree

Rettangolo

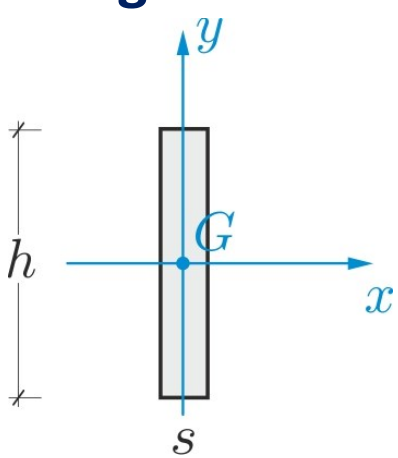


$$I_x = \frac{1}{12} b h^3$$

$$I_y = \frac{1}{12} h b^3$$

$$\boxed{I_{xy} = 0}$$

Rettangolo sottile ($s \ll h$)



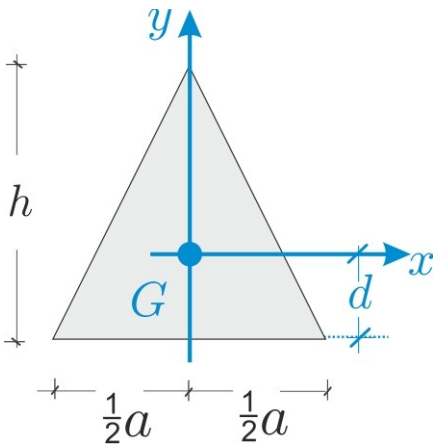
$$I_x = \frac{1}{12} s h^3$$

$$I_y = \frac{1}{12} h s^3 \cong 0$$

$$\boxed{I_{xy} = 0}$$

1. Geometria delle aree

Triangolo isoscele



$$A = \frac{1}{2}ah$$

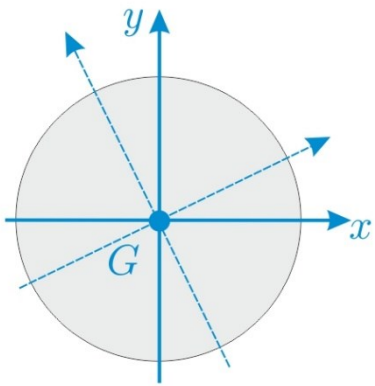
$$d = \frac{1}{3}h$$

$$I_x = \frac{1}{36}ah^3$$

$$I_y = \frac{1}{48}ha^3$$

$$\boxed{I_{xy} = 0}$$

Cerchio di raggio R



$$A = \pi R^2$$

$$I_x = I_y = \frac{1}{4}\pi R^4$$

$$I_r = \frac{1}{2}\pi R^4$$

$$\boxed{I_{xy} = 0}$$